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An Extended Approach to the Field-Theoretical Time Operators

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Abstract

It is proved that within the two- or three-dimensional formulations of quantum field theory the possibility exists of performing the time description in the whole Hilbert space of the particle and of the antiparticle states respectively. In contrast to the Klein-Gordon field, the present time description of the whole Dirac particle–antiparticle field may be adequately performed only in the one-dimensional case.

1. *Introduction*

The definition of the field-theoretical time operators previously performed (Papp, 1974) needs additional discussion. Whereas the space operator is unequivocally defined, one may define—in agreement with the correspondence principle—a set of three field-theoretical time operators T_i corresponding to the classical expression x_i/v_i, $i = 1, 2, 3$. In this way, we obtain three time operators which describe the time measurements performed in respect to the projection of the free motion on the coordinate axis respectively. It has been generally proved—in contrast to the classical description—that in the relativistic case all the three directions cannot be simultaneously maintained in order to perform the time measurement. In these conditions a certain direction has to be chosen, so that the time-measuring process concern only the free motion projected on that direction. In this sense, the requirement of a one-dimensional field also expresses a consequence of the manifestly one-dimensional form of the starting free-motion time expression x_i/v_i.

Consequently, in order to perform a more general time description, another more suitable classical free-motion time expression has to be chosen. Indeed, when a certain direction is needed in order to perform a time measurement, it is in the spirit of quantum mechanics to choose it adequately. In these conditions it is quite natural to choose as 'time-measuring' direction that of the momentum. In this way we obtain the time operator already proposed by Lippmann (1966), i.e. the quanto-mechanical time operator which corresponds to the classical expression r_p/v where $r_p = (1/p)(\mathbf{p}\,.\,\mathbf{r})$.

In the following the field-theoretical counterparts of the free-motion time r_p/v will be calculated and discussed. Ultimately, in the two- or three-dimensional case, the vanishing condition of the annihilation (or creation) operators at the origin is not necessarily implied. The calculations will be performed both in the relativistic and non-relativistic cases. The same notation as previously will be maintained.

2. *The Non-Relativistic Time Operator*

Starting from the quanto-mechanical time operator

$$\hat{t}_p = m_0 \mathbf{r}\,.\widehat{\left(\frac{\mathbf{p}}{p^2}\right)} \tag{2.1}$$

and defining the field-theoretical time operator as

$$T_p(t) = \int d\mathbf{x}\,\psi^*(\mathbf{x}, t)\hat{t}_p\psi(\mathbf{x}, t) \tag{2.2}$$

we obtain the binary operator

$$T_p(t) = \int d\mathbf{p}\,a^*(\mathbf{p})\left[im_0\frac{\mathbf{p}}{p^2}\cdot\frac{\partial}{\partial\mathbf{p}} + t - i\frac{m_0}{p^2}\right]a(\mathbf{p}) \tag{2.3}$$

when the expressions

$$\lim_{p_i\to 0} p_i \int dp_j\,dp_k\,\frac{1}{p^2}a^*(\mathbf{p})a(\mathbf{p}) < \infty \qquad (i \neq j \neq k) \tag{2.4}$$

are bounded in the weak sense. Taking for example $p_1 = \epsilon \to 0$ and choosing the cylindrical coordinates (p_1, q, φ), where $q = \sqrt{(p_2^2 + p_3^2)}$, it may be proved that

$$\mathscr{F}(q_1, \epsilon) \equiv \left\| \int_0^{q_1}\int_0^{2\pi} q\,dq\,d\varphi\,\frac{1}{q^2+\epsilon^2}a^*a \right\| \lesssim \pi M^2\frac{\epsilon^2}{q_1^2} + 2\pi M^2 \ln\frac{q_1}{\epsilon} \tag{2.5}$$

as soon as $q_1 > \epsilon$ and $\|a(\mathbf{p})\| \leqslant M$ in the vicinity of the origin, where the convergence in the norm is now considered. Consequently.

$$\lim_{\epsilon\to 0} \epsilon\mathscr{F}(q_1, \epsilon) = 0 \tag{2.6}$$

so that in order to fulfil the relations (2.4) it is sufficient to require the annihilation operator to be bounded at the origin. In the two-dimensional case it may similarly be proved that

$$\left\| \int_{-q_1}^{q_1} dq \frac{1}{q^2+\epsilon} a^* a \right\| \leqslant \frac{2M^2}{\epsilon} \operatorname{arc\,tg} \frac{q_1}{\epsilon} \simeq \frac{2\pi M^2}{\epsilon} \tag{2.7}$$

so that the above conclusions also maintain their validity in the present situation. We may thus conclude that in the above cases the quanto-mechanical time description may be performed—in contrast to the one-dimensional case—in the whole Hilbert space of the physical states.

3. *The Relativistic Time Operators*

The field-theoretical counterparts

$$T_p(t) = i \int d\mathbf{x} \Phi^{(+)*}(\mathbf{x}, t) \overleftrightarrow{\partial}_t \hat{t}_p \Phi^{(+)}(\mathbf{x}, t) \tag{3.1}$$

and

$$T_p^{(0)}(t) = \int d\mathbf{x} \psi^{(+)*}(\mathbf{x}, s, -s, t) \hat{t}_p \psi^{(+)}(\mathbf{x}, s, -s, t) \tag{3.2}$$

of the quanto-mechanical time operator

where

$$\psi^{(+)}(\mathbf{x}, s, -s, t) \equiv \chi(\mathbf{x}, t),$$

$$\hat{t}_p = \mathbf{r} . \widehat{\left(\mathbf{p} \frac{p_0}{p^2}\right)}, \qquad (p^2 \equiv \mathbf{p}^2) \tag{3.3}$$

are given by the binary time operators

$$T_p(t) = \int d\mathbf{p}\, a^{(+)*}(\mathbf{p}) \left[i p_0 \frac{\mathbf{p}}{p^2} \cdot \frac{\partial}{\partial \mathbf{p}} + t - i \frac{m_0^2}{p^2 p_0} \right] a^{(+)}(\mathbf{p}) \tag{3.4}$$

and

$$T_p^{(0)}(t) = \sum_{\pm s} \int d\mathbf{p}\, b^*(\mathbf{p}, s) \left[i p_0 \frac{\mathbf{p}}{p^2} \cdot \frac{\partial}{\partial \mathbf{p}} + t - i \frac{m_0^2}{p^2 p_0} \right] b(\mathbf{p}, s) \tag{3.5}$$

for the Klein-Gordon and Dirac positive-energy fields respectively. In order to define the time operator (3.5) the spinorial relation

$$u^*(\mathbf{p}, s') \mathbf{p} . \frac{\partial}{\partial \mathbf{p}} u(\mathbf{p}, s) = \frac{p^2}{2 m_0 p_0} \delta_{s's} \tag{3.6}$$

which is valid irrespective of the spatial dimension, has been used. The boundary conditions are analogous to the non-relativistic ones, so that the time operators above also act in the whole Hilbert space.

Performing the expansion after spherical harmonics of the Klein-Gordon annihilation operator:

$$a^{(+)}(\mathbf{p}) = p^{-1} \sum_{l,m} a_{l,m}^{(+)}(p)\, Y_{l,m}(\text{vers}\, \mathbf{p} \,.\, \mathbf{k}) \tag{3.7}$$

where $\mathbf{k}$ expresses the unit vector of the Op_3 axis, we obtain

$$T_p(t) = \sum_{l,m} \int_0^\infty dp\, a_{l,m}^{(+)*}(p) \left[i \frac{\partial}{\partial p_0} + t - i \frac{m_0^2}{p^2 p_0} \right] a_{l,m}^{(+)}(p) \tag{3.8}$$

where the new annihilation operators $a_{l,m}^{(+)}(p)$ have to vanish at the origin in order to assure not only a bounded value of the $a^{(+)}(\mathbf{p})$ operator in that region, but also the existence of the hermitic time operator. It now becomes clear that the time description performed in respect to the given value the angular momentum and also to its projection on the Op_3 axis is a special case of the time description performed in the three-dimensional case. In this respect we also have to mention that generally the l,m- summation cannot be inverted with the $\mathbf{p}$-momentum and coordinate integrations, i.e. it is generally not uniformly convergent. This fact also explains why the above definition of the time operator—which has been performed when the inversion of the summation with the integrations is (implicitly) assumed—is valid for any t-value, whereas the definition previously analysed (Papp, 1974) is generally valid only for large t-values. Indeed, in contrast to the above situation, in the previous case the well-defined value of the angular momentum is considered from the very beginning. On the other hand the $\hat{r}_p$-operator is now a binarily one for $j \neq 1$.

Performing the calculations it may be proved that the time description may be similarly performed with respect to the two- or three-dimensional Klein-Gordon antiparticle field, and also with respect to the whole Klein-Gordon particle-antiparticle field. Whereas the time description of the Dirac antiparticle field may be performed similarly to that of the Dirac particle field, the present time description of the whole Dirac field may be adequately performed only with respect to the one-dimensional field. Indeed, in the two- or three-dimensional case the spinorial relations imply interference contributions due to the spin.

It may now be remarked that the only space operator which can be adequately defined with respect to the two- or three-dimensional Dirac particle (or antiparticle) field is the field theoretical counterpart of the space projection operator on the $\mathbf{p}$-momentum

$$r_p = \mathbf{r} \,.\, \widehat{\left(\frac{\mathbf{p}}{p}\right)} \tag{3.9}$$

which is canonically conjugate to the p-momentum. Considering for example the Dirac particle field, we obtain the binary operator

$$R_p^{(0)}(t) = \sum_{\pm s} \int d\mathbf{p}\, b^*(\mathbf{p}, s) \left[i \frac{\mathbf{p}}{p} \cdot \frac{\partial}{\partial \mathbf{p}} + t \frac{p}{p_0} + i \frac{j-1}{2p} \right] b(\mathbf{p}, s) \quad (3.10)$$

where $j = 1, 2, 3$, is the spatial dimension of the field and where the space imprecision is given by $(j-1)\langle 1/2p \rangle$.

The field-theoretical counterpart of the space operator (3.9) may be also written without any restriction, for the Klein–Gordon (particle, antiparticle and particle-antiparticle) field. Thus the space operator of the Klein–Gordon particle field is given by

$$R_p(t) = \int d\mathbf{p}\, a^{(+)*}(\mathbf{p}) \left[i \frac{\mathbf{p}}{p} \cdot \frac{\partial}{\partial \mathbf{p}} + t \frac{p}{p_0} + i \frac{j-1}{2p} \right] a^{(+)}(\mathbf{p}) \qquad (3.11)$$

where the space imprecision is $(j-1)\langle 1/2p \rangle$. The hermitian space-time operators may be obtained in the usual manner.

4. *Conclusions*

The one-dimensional approach (performed along an arbitrary direction) to the time description previously analysed has been extended in order to unequivocally include the two- and three-dimensional cases.

For this purpose a more suitable form of the classical free-motion time is needed. Reducing the above time description to the one of a one-dimensional field, the previous boundary conditions are implied. Consequently, within the field-theoretical descriptions there is no general time description definable, irrespective of the spatial dimension of the field, in the whole Hilbert space. Otherwise the one-dimensional field description has to be ignored and this fact would contradict the consistency requirements.

References

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